NONLINEAR POWER-LAW RESISTANCE FLUID FILTRATION THROUGH POROUS SEMISPACE WITH A GROOVE

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A method is proposed for solving some planar problems of nonlinear filtration with power-law resistance in the case of straight-line boundaries of the filtration region.

Planar fluid filtration with nonlinear resistance law has been the subject of investigations in a number of papers. One method which is very effective is the time curve method of Chaplygin which employs linearization of the equations [1]-[4]. However, even when linearization had been used it was rather difficult to obtain exact solutions in the general case.

A method is presented in the present work for solving nonlinear filtration problems for power-law resistance based on the Chaplygin transformation. The main points of the method as well as its potential are demonstrated on the example of a classical filtration problem through a porous semispace with a groove.

The stationary fluid motion in a porous body for power-law resistance is described by the system

$$\frac{\partial\tilde{\psi}}{\partial\beta} = \frac{\sqrt{n+1}}{\chi}\exp(-2\varepsilon\tau)\frac{\partial\tilde{p}}{\partial\tau},$$
$$\frac{\partial\tilde{\psi}}{\partial\tau} = -\frac{\sqrt{n+1}}{\chi}\exp(-2\varepsilon\tau)\frac{\partial\tilde{p}}{\partial\beta}. \quad (1)$$

By eliminating $\tilde{\psi}$ from (1) and by the change of the variables

$$\tilde{p} = Q\exp(\varepsilon\tau), \quad (2)$$

the Helmholtz equation

$$\frac{\partial^2 Q}{\partial\beta^2} + \frac{\partial^2 Q}{\partial\tau^2} - \varepsilon^2 Q = 0 \quad (3)$$

is obtained for the filtration under consideration. The relation between the variables τ and β and the physical coordinates is given by the equations

$$d\tilde{x} = -\frac{1}{\chi}\exp(-\sqrt{n+1}\,\tau)\cos\beta d\tilde{p} - \exp\left(-\frac{\tau}{\sqrt{n+1}}\right)\sin\beta d\tilde{\psi},$$
$$\partial\tilde{y} = -\frac{1}{\chi}\exp(-\sqrt{n+1}\,\tau)\sin\beta d\tilde{p} + \exp\left(-\frac{\tau}{\sqrt{n+1}}\right)\cos\beta d\tilde{\psi}. \quad (4)$$

A fluid filtration is now considered through a planar porous semispace with a groove BA (Fig. 1a) of height d. Let the pressure be p_1 = const on the horizontal half-line CB and p = 0 on the half-line BD. At the points C and D at infinity one has $\tau = -\infty$ and one has $\tau = \infty$ at the point A. The filtration region for the coordinate system xBy, adopted in the Fig. 1a in the variables τ and β, is given by the infinite strip $0 \le \beta \le \pi$ (Fig. 1b). Since in the problem under consideration the symmetry of the flow lines with respect to the Bx axis is obvious therefore the velocity is the same at the points B_+ and B_-. Suppose that to this

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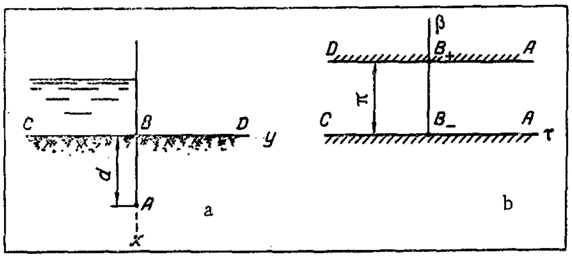

Fig. 1. Fluid filtration diagram through planar porous semispace.

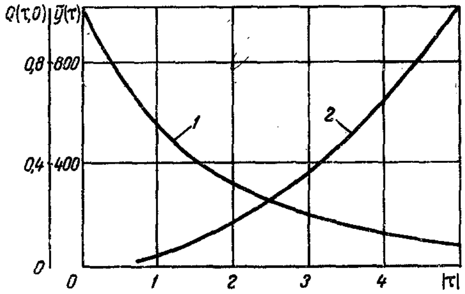

Fig. 2. $Q(\tau, 0)$ (1) and $y(\tau, \pi)$ (2) as functions of τ.

velocity v_0 there corresponds $\tau = 0$. The expression $Q(\tau, 0) = \exp(-\varepsilon\tau)$ on the straight line CB tends to infinity for $\tau \to -\infty$ but it is $Q(\tau, 0)\exp(b\tau) \to 0$ for $b > \varepsilon$. It is obvious that $Q(\tau, \beta)\exp(b\tau)$ approaches 0 for $\tau \to -\infty$, $Q(\tau, \beta) \to 0$ for $\tau \to \infty$.

One represents $Q(\tau, \beta)$ in the form of a generalized Fourier integral

$$Q(\tau, \beta) = \frac{1}{2\pi} \int_{-\infty+ib}^{\infty+ib} \exp(i\lambda\tau)\,\overline{Q}_-(\lambda, \beta)\,d\lambda + \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\lambda\tau)\,\overline{Q}_+(\lambda, \beta)\,d\lambda, \tag{5}$$

where

$$\overline{Q}_-(\lambda, \beta) = \int_{-\infty}^{0} \exp(-i\lambda\xi)\,Q(\xi, \beta)\,d\xi,$$

$$\overline{Q}_+(\lambda, \beta) = \int_{0}^{\infty} \exp(-i\lambda\xi)\,Q(\xi, \beta)\,d\xi.$$

By inserting (5) in (3) and introducing the operator notation $L(\;) = [d^2(\;)/d\beta^2] - q^2(\;)$ one finds that

$$\frac{1}{2\pi} \int_{-\infty+ib}^{\infty+ib} \exp(i\lambda\tau)\,L(\overline{Q}_-)\,d\lambda + \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\lambda\tau)\,L(\overline{Q}_+)\,d\lambda = 0.$$

It is not difficult to see that $L(\overline{Q}_-)$ is regular in some upper half-plane λ and $L(\overline{Q}_+)$ in a lower half-plane. Therefore everywhere one has

$$\frac{d^2\overline{Q}}{d\beta^2} - q^2\overline{Q} = 0, \tag{6}$$

where $\overline{Q} = \overline{Q}_+(\lambda, \beta) + \overline{Q}_-(\lambda, \beta)$. The solution of (6) is given by

$$\overline{Q}(\lambda, \beta) = \overline{Q}(\lambda, 0)\frac{\operatorname{sh} q(\pi - \beta)}{\operatorname{sh} q\pi} + \overline{Q}(\lambda, \pi)\frac{\operatorname{sh} q\beta}{\operatorname{sh} q\pi}, \tag{7}$$

where

$$\overline{Q}(\lambda, 0) = \overline{Q}_+(\lambda, 0) + \frac{i}{\lambda - i\varepsilon}; \quad \overline{Q}(\lambda, \pi) = \overline{Q}_+(\lambda, \pi).$$

To find the unknowns $\overline{Q}_+(\lambda, 0)$ and $\overline{Q}_+(\lambda, \pi)$ the boundary condition on the left- and right-hand side of the groove is used, namely

$$\left.\frac{\partial Q(\tau, \beta)}{\partial \beta}\right|_{\beta=0} = 0, \quad \tau > 0;$$
$$\left.\frac{\partial Q(\tau, \beta)}{\partial \beta}\right|_{\beta=\pi} = 0, \quad \tau > 0. \tag{8}$$

It follows from (8) that $\left.\frac{d\bar{Q}}{d\beta}\right|_{\beta=0} = \bar{\Psi}_{1-}(\lambda), \left.\frac{d\bar{Q}}{d\beta}\right|_{\beta=\pi} = \bar{\Psi}_{2-}(\lambda)$, where $\Psi_{1-}(\lambda)$ and $\bar{\Psi}_{2-}(\lambda)$ are regular in some upper half-plane of the complex variable λ. One finds with the aid of (7) and (8) that

$$[\bar{Q}(\lambda, 0) - \bar{Q}_+(\lambda, \pi)]\, q \operatorname{cth} \frac{q\pi}{2} = -\bar{\Psi}_{2-}(\lambda) - \bar{\Psi}_{1-}(\lambda), \tag{9}$$

$$[\bar{Q}(\lambda, 0) + \bar{Q}_+(\lambda, \pi)] \operatorname{th} \frac{q\pi}{2} = \bar{\Psi}_{2-}(\lambda) - \bar{\Psi}_{1-}(\lambda). \tag{9'}$$

It follows from the symmetry of the flow-lines that (9') can be replaced by

$$\bar{Q}(\lambda, 0) + \bar{Q}_+(\lambda, \pi) = 0. \tag{10}$$

The system (9) and (10) is investigated using the Wiener–Hopf method [5]. The notation

$$q \operatorname{cth} \frac{q\pi}{2} = K(\lambda),$$

is introduced and this expression is factorized:

$$K(\lambda) = \varphi_+(\lambda)\varphi_-(\lambda),$$

where $\varphi_+(\lambda)$ is regular and has no zeros in the lower half-plane and $\varphi_-(\lambda)$ has the same properties in the upper half-plane. By using the infinite-product expansions for $\sinh q\pi/2$ and $\cosh q\pi/2$ and in view of the fact that the series $\sum_{k=1}^{\infty}\left(\frac{1}{s_k} - \frac{1}{r_k}\right)$ is convergent one can omit the corresponding exponential Weierstrass factors and one finds

$$\varphi_+(\lambda) = \prod_{k=1}^{\infty} \frac{2k}{2k-1} \cdot \frac{\lambda - is_k}{\lambda - ir_k},$$
$$\varphi_-(\lambda) = \frac{2}{\pi} \prod_{k=1}^{\infty} \frac{2k}{2k-1} \cdot \frac{\lambda + is_k}{\lambda + ir_k}. \tag{11}$$

It will be necessary in our subsequent considerations to have estimates of the behavior of these functions in their respective regularity half-planes. Starting directly with the definition of $K(\lambda)$ it is established that on the real axis both these functions are of the order $\sqrt{\lambda}$.

The function

$$\frac{\pi}{2}\varphi_-(\lambda) = \prod_{k=1}^{\infty} \frac{2k}{2k-1} \cdot \frac{\lambda + is_k}{\lambda + ir_k}$$

is now investigated in the remainder of the regularity half-plane. Denoting the common term of the infinite product by a_k and setting $\lambda = Re^{i\theta}$ one finds that

$$|a_k| = \frac{2ks_k}{(2k-1)r_k} \cdot \frac{\sqrt{1 + 2\frac{R}{s_k}\sin\theta + \frac{R}{s_k^2}}}{\sqrt{1 + 2\frac{R}{r_k}\sin\theta + \frac{R^2}{r_k^2}}} = \frac{2ks_k\sqrt{(s_k)}}{(2k-1)r_k\sqrt{(r_k)}}.$$

The logarithmic derivative of $|a_k|$ with respect to θ is

$$R\cos\theta\left[\frac{1}{s_k(s_k)} - \frac{1}{r_k(r_k)}\right].$$

The latter is positive in the upper half-plane for a motion in the direction of the imaginary axis; hence it follows that $|\varphi_-(\lambda)|$ increases. On the imaginary axis $\theta = \pi/2$ and

$$\frac{d \ln |a_k|}{dR} = \frac{1}{s_k + R} - \frac{1}{r_k + R} > 0.$$

Therefore on the imaginary axis $|\varphi_-(\lambda)|$ also increases monotonically in the upper half-plane. The order of magnitude of this increase is found by setting $\lambda = is_j$. Then

$$\frac{\pi}{2}\varphi_-(is_j)=\prod_{k=1}^{\infty}\frac{2k}{2k-1}\cdot\frac{s_j+s_k}{s_j+r_k}=\prod_{k=1}^{j}\frac{2k}{2k-1}\cdot\frac{s_j+s_k}{s_k+r_k}\prod_{k=j+1}^{\infty}\frac{2k}{2k-1}\cdot\frac{s_j+s_k}{s_j+r_k}.$$

In the absolutely convergent infinite product

$$\prod_{k=j+1}^{\infty}\frac{2k}{2k-1}\cdot\frac{s_j+s_k}{s_j+r_k}=\prod_{\nu=1}^{\infty}a_\nu \qquad (\nu=k-j)$$

each of the factors is greater than unity; therefore, the product itself must exceed unity for any j. With j increasing a_ν decreases monotonically and the entire product decreases monotonically having as its limit for $j \to \infty$ some value $M \geq 1$ (it follows from our subsequent considerations that $M > 1$). Consequently, the order of magnitude of the increase of $\varphi_-(\lambda)$ on the imaginary axis is determined by the product

$$\prod_{k=1}^{j}\frac{2k}{2k-1}\cdot\frac{s_j+s_k}{s_j+r_k}.$$

The general term of this product is

$$a_{kj}=\frac{2k}{2k-1}\cdot\frac{s_j+s_k}{s_j+r_k}<\frac{2k}{2k-1}.$$

Therefore, $(\pi/2)\varphi_-(is_j)$ does not increase more rapidly on the imaginary axis [6] than

$$\Pi_j=M\prod_{k=1}^{j}\frac{2k}{2k-1}=M2^{2j}\frac{(j!)^2}{(2j)!}\approx M\sqrt{\frac{\pi}{2}}\sqrt{s_j}.$$

Since $\varphi_-(iR)$ is monotonic it follows that this is also valid for any R. Employing now the previous estimates it is established that one has

$$|\varphi_-(\lambda)|=A(\lambda)\,|\sqrt{\lambda}\,|,$$

on the entire upper half-plane where $A(\lambda) > \varkappa > 0$ is bounded ($\varkappa$ = const). The estimate thus obtained is also valid for $\varphi_+(\lambda)$ in the lower half-plane.

By substituting the factorized $K(\lambda)$ in (9) one finds that

$$\overline{Q}_+(\lambda,\,0)+\overline{Q}_+(\lambda,\,\pi)=-\frac{i}{\lambda-i\varepsilon}, \tag{12}$$

$$\left[\overline{Q}_+(\lambda,\,0)-\overline{Q}_+(\lambda,\,\pi)\right]\varphi_+(\lambda)+\frac{i\varphi_+(\lambda)}{\lambda-i\varepsilon}=-\frac{\overline{\Psi}_{2-}(\lambda)-\overline{\Psi}_{1-}(\lambda)}{\varphi_-(\lambda)}. \tag{13}$$

When the obvious behavior of $\tilde{p}(\tau, \beta)$ for $\tau \to \pm\infty$ has been taken into account one can establish without any difficulty that the left- and right-hand sides of (13) are regular in the entire λ-plane with the exception of the pole at the point $\lambda = i\varepsilon$. At this point the Laurent expansion of these functions is valid for the entire plane. Then in view of the obtained estimates for the function from (11) the regular part of the expansion can be expressed by a polynomial which is identically equal to zero since for $\lambda \to \infty$ the right-hand side of (13) approaches 0 in the upper half-plane.

At the point $\lambda = i\varepsilon$ the pole can only be simple since otherwise the originals for the Fourier transformation would contain components with a multiplier τ^m (m a positive integer); this is not possible from the physical considerations since the pressure is finite in the entire filtration region.

Hence by using (12) and (13) one finds that

$$\overline{Q}_+(\lambda,\,0)=-\frac{i}{\lambda-i\varepsilon}+\frac{ia}{(\lambda-i\varepsilon)\varphi_+(\lambda)},$$

$$\overline{Q}_+(\lambda,\ \pi) = -\frac{ia}{(\lambda - i\varepsilon)\varphi_+(\lambda)}, \tag{14}$$

where a is a real parameter.

One can determine $Q_+(\tau,\ \pi)$ by using the residue theorem for the poles of $\overline{Q}_+(\lambda,\ \pi)$ in the upper half-plane of the variable λ:

$$Q_+(\tau,\ \pi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(i\lambda\tau)\overline{Q}_+(\lambda,\ \pi)\,d\lambda = a\left\{\varkappa \exp(-\varepsilon\tau) - \frac{4}{\pi^2}\sum_{j=1}^{\infty} \frac{\exp(-s_j\tau)}{s_j(s_j-\varepsilon)\Phi(s_j)}\right\}. \tag{15}$$

The condition $Q_+(0,\ \pi)$ is satisfied for the just found (15) for any a; in this case the Fourier integral (15) can be evaluated by considering only the lower-half plane of λ in which the subintegral function has no singularities.

One also finds from (14) that

$$Q_+(\tau,\ 0) = \exp(-\varepsilon\tau) - a\left\{\varkappa \exp(-\varepsilon\tau) - \frac{4}{\pi^2}\sum_{j=1}^{\infty} \frac{\exp(-s_j\tau)}{s_j(s_j-\varepsilon)\Phi(s_j)}\right\}. \tag{16}$$

The parameter a can be determined from the obvious condition

$$\lim_{\tau\to\infty}\left[Q_+(\tau,\ 0)\exp(\varepsilon\tau)\right] = \lim_{\tau\to\infty}\left[Q_+(\tau,\ \pi)\exp(\varepsilon\tau)\right] = \frac{1}{2}. \tag{16'}$$

By using (15) or (16') one finds that $a = 1/2\varkappa$.

Finally, one obtains

$$Q_+(\tau,\ \pi) = \frac{1}{2}\exp(-\varepsilon\tau) - \frac{2}{\pi^2\varkappa}\sum_{j=1}^{\infty}\frac{\exp(-s_j\tau)}{s_j(s_j-\varepsilon)\Phi(s_j)},$$

$$Q_+(\tau,\ 0) = \frac{1}{2}\exp(-\varepsilon\tau) + \frac{2}{\pi^2\varkappa}\sum_{j=1}^{\infty}\frac{\exp(-s_j\tau)}{s_j(s_j-\varepsilon)\Phi(s_j)}.$$

The first expression (4) is used to find the filtration parameter χ; integrating it for $\beta = 0$ with respect to τ between the limits 0 and ∞ and using (16) one finds:

$$\chi = \frac{2}{\pi^2\varkappa}\left\{\sum_{j=1}^{\infty}\frac{1}{\left(s_j + \dfrac{n+2}{2\sqrt{n+1}}\right)(s_j-\varepsilon)\Phi(s_j)} - \varepsilon\sum_{j=1}^{\infty}\frac{1}{s_j\left(s_j + \dfrac{n+2}{2\sqrt{n+1}}\right)(s_j-\varepsilon)\Phi(s_j)}\right\}. \tag{17}$$

For the case of $n = 1$ (quadratic filtration) it was found from (17) that $\chi \cong 0.17$.

From (12), (13), and (14) one obtains that

$$\overline{\Psi}_{z-}(\lambda) = \left.\frac{d\overline{Q}(\lambda,\ \beta)}{d\beta}\right|_{\beta=\pi} = -\frac{i}{2\varkappa(\lambda - i\varepsilon)}\varphi_-(\lambda). \tag{18}$$

By applying the residue theorem to the expression (18) one obtains

$$\left.\frac{\partial Q(\tau,\ \beta)}{\partial\beta}\right|_{\beta=\pi} = -\frac{1}{\pi}\exp(-\varepsilon\tau) - \frac{4}{\pi\varkappa}\sum_{j=1}^{\infty}\frac{j^2\Phi(r_j)}{r_j(r_j+\varepsilon)}\exp(r_j\tau) \tag{19}$$

$$(\tau < 0).$$

Using the second expression in (4) as well as (1), (19), and (2) one obtains

$$\bar{y} = \frac{1}{\pi\chi}\left[\exp(-\sqrt{n+1}\,\tau) - 1\right]$$

$$-\frac{4\sqrt{n+1}}{\pi\varkappa\chi}\sum_{j=1}^{\infty}\frac{j^2\Phi(r_j)}{r_j(r_j+\varepsilon)\left(r_j-\frac{n+2}{2\sqrt{n+1}}\right)}\left\{\exp\left[\left(r_j-\frac{n+2}{2\sqrt{n+1}}\right)\tau\right]-1\right\}. \quad (20)$$

The formula (20) describes the distribution of the velocities of setting down along the right-hand boundary of a porous half-space where $\tau < 0$.

The graphs of $Q = Q_+(\tau, 0)$ and $\tilde{y} = \tilde{y}(\tau, \pi)$ for the case of quadratic filtration are shown in Fig. 2.

NOTATION

τ, β are the Chaplygin variables;

$\tau = \sqrt{n+1}\ln v/v_0$;

$\psi = \tilde{\psi}/v_0 d$ is the dimensionless flow function;

$\tilde{p} = p/p_1$ is the dimensionless pressure;

d, p_1 are the characteristic dimension and pressure;

$\chi = v_0^{n+1}d/p_1\alpha_1$ is the dimensionless filtration parameter;

α_1 is the constant characterizing porous medium and fluid;

$n + 1$ is the degree of filtration (at $n = 0$ filtration is linear);

$\tilde{x} = x/d$, $\tilde{y} = y/d$;

λ is the Fourier parameter;

$q = \sqrt{\lambda^2 + \varepsilon^2}$;

$s_k = \sqrt{(2k-1)^2 + \varepsilon^2}$;

$r_k = \sqrt{4k^2 + \varepsilon^2}$;

$\varepsilon = n/\sqrt{n+1}$;

$$\Phi(z) = \prod_{k=1}^{\infty}\frac{2k-1}{2k}\cdot\frac{z+r_k}{z+s_k};$$

$$\varkappa = \frac{1}{\varphi_+(i\varepsilon)} = \prod_{k=1}^{\infty}\frac{2k-1}{2k}\cdot\frac{r_k-\varepsilon}{s_k-\varepsilon} = \Phi(-\varepsilon).$$

LITERATURE CITED

1. S. A. Khristianovich, Prikl. Matem. i Mekh., 4, No. 1 (1940).
2. F. Engelund, Trans. Danish Acad. Tech. Sci., No. 3 (1953).
3. V. M. Entov, Prikl. Matem. i Mekh., No. 5 (1967).
4. V. M. Entov, Prikl. Matem. i Mekh., No. 3 (1968).
5. E. Titchmarsh, Introduction to the Theory of Fourier Integrals [Russian translation], Gostekhizdat, Moscow (1948).
6. N. N. Lebedev, Special Functions and Their Applications [in Russian], Fizmatgiz, Moscow (1963).